

# Light Meson Decay Constants

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## Abstract

We estimate the decay constants  $f_\pi$  and  $f_K$  of the light mesons within a relativistic quantum-field model of interacting quarks and gluons confined analytically. The necessary physical parameters, the quark masses  $m_u$  and  $m_s$ , the coupling constant  $\alpha_s$  and the confinement scale  $\Lambda$  have been obtained from our previous investigation on the meson ground states, orbital and radial excitations by using the ladder Bethe-Salpeter equation. Our model provides a solid framework to compute the meson spectra, the lowest glueball state and decay constants of light mesons from the basic principles of QCD and QFT.

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## 1 Introduction

The pseudoscalar mesons, especially the  $\pi$  and  $K$ , are studied extensively to understand the under-structures from nonperturbative QCD (e.g., [1]). The decay constants of the light mesons are one of the basic parameters in the particle physics and form an extremely important test ground for any nonperturbative QCD methods. Particularly, the pseudoscalar-meson decay constants  $f_P$  play an important role in hadron phenomenology (e.g., the leptonic decay width is proportional to  $f_P^2$  and the  $V_{us}$  element is extracted from the ratio  $f_K/f_\pi$ ) and can be used as a test ground for models.

A number of theoretical approaches obeying an accuracy comparable to the uncertainties in the experimental data, has relied on models containing too many, in some cases even nonphysical, parameters. On the other hand, the scaling property of QCD implies that all characteristics of hadrons must depend on a few global parameters.

Among the significant theoretical approaches, the Bethe-Salpeter equation (BSE) is a conventional approach in dealing with the two-body relativistic bound state problems [2] and its solutions give useful information about the under-structure of the mesons. Many analytical and numerical calculations indicate that the ladder BSE with phenomenological models can give model independent results successful descriptions of the long distance properties of the low energy QCD and the QCD vacuum [3, 4, 1].

Below we adopt a relativistic quantum field model of interacting quarks and gluons under the analytic confinement, solve the BSE in the one-gluon exchange approximation for the quark-antiquark bound states and estimate the decay constants  $f_\pi$  and  $f_K$ . The model relies on only necessary physical parameters  $\{m_f, \alpha_s, \Lambda\}$  which have been fixed to fit the meson masses in the ground states, orbital and radial excitations as well as the lowest-state glueball mass [5, 6].

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## 2 Two-quark Bound States

In the conventional quark model mesons are  $(q\bar{q}')$  bound states of quarks and antiquarks (the flavors may be different). We deal with the pseudoscalars  $P$  ( $J^{PC} = 0^{-+}$ ) and vectors  $V$  ( $1^{--}$ ), the most established sectors of hadron spectroscopy.

Consider a Yukawa-type interaction of quarks and gluons with the Lagrangian

$$\mathcal{L} = (\bar{\Psi} S^{-1} \Psi) + \frac{1}{2} (\phi D^{-1} \phi) + g (\bar{\Psi}_\alpha^i i \gamma_\mu^{\alpha\beta} t_{ij}^a \Psi_\beta^j \phi_\mu^a), \quad (1)$$

where  $\Psi_{\alpha f}^a(x)$  is the quark field,  $\phi_\mu^a(x)$  is the gluon field,  $g$  is the coupling strength,  $t^a$  are the Gell-Mann matrices and  $\{a, \alpha, f\}$  are the color, spin and flavor indices.

The conventional QCD encounters a difficulty by defining the explicit quark and gluon propagator at the confinement scale [7]. Particularly, the infrared form of the gluon propagator is still a controversial aspect [8], there are various versions, with different behaviors for the infrared region: finite, zero and more divergent than  $1/k^2$ , each one with its advantages and inconveniences [9, 10, 11, 12].

Nevertheless, the matrix elements of hadron processes at large distance are in fact integrated characteristics of the quark and gluon functions and taking into account the correct global symmetry properties and their breaking by introducing additional physical parameters may be more important than the working out in detail (e.g., [13, 4, 14]). Besides, the bound states of quarks and gluons may be found as the solution of the BSE in a variational-integral form [5, 6] that is low sensible on tiny details of integrands.

We consider the simplest effective quark and gluon propagators:

$$\begin{aligned} \tilde{S}_{\alpha\beta}^{ij}(\hat{p}) &= \delta^{ij} \frac{\{i\hat{p} + m [1 + \gamma_5 \omega(m)]\}_{\alpha\beta}}{m^2} \exp \left\{ -\frac{p^2 + m^2}{2\Lambda^2} \right\}, \\ D_{ab}^{\mu\nu}(x) &= \delta_{ab} \delta^{\mu\nu} \frac{\Lambda^2}{(4\pi)^2} \exp \left\{ -\frac{x^2 \Lambda^2}{4} \right\} = \delta_{ab} \delta^{\mu\nu} D(x), \end{aligned} \quad (2)$$

where  $\hat{p} = p_\mu \gamma_\mu$ ,  $m$  - the quark mass and  $0 < \omega(m) < 1$  [6]. These entire analytic functions in the Euclidean metric keep the spirit of the *analytic confinement* [15, 16, 4] and imply that each isolated quark and gluon is confined in the background gluon field.

The meson mass is derived from [6]:

$$1 + \lambda_{\mathcal{N}}(M_{\mathcal{N}}^2) = 0. \quad (3)$$

where the two-quark bound states are obtained by diagonalizing the quadratic part in the meson partition function on the full orthonormal system  $\{U_{\mathcal{N}}\}$ . This is nothing else but the solution of the appropriate BSE (in the one-gluon approximation):

$$\iint dx dy U_{\mathcal{N}}(x) \left\{ 1 + g^2 \sqrt{D(x)} \Pi_{JJ'}(p^2, x, y) \sqrt{D(y)} \right\} U_{\mathcal{N}'}(y) = \delta^{\mathcal{N}\mathcal{N}'} [1 + \lambda_{\mathcal{N}}(-p^2)], \quad (4)$$

where, the two-point function is introduced:

$$\Pi_{JJ'}(p^2, x, y) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} \mathbf{Tr} \left[ \Gamma_J \tilde{S}(\hat{k} + \mu_1 \hat{p}) \Gamma_{J'} \tilde{S}(\hat{k} - \mu_2 \hat{p}) \right]$$

and  $\Gamma_J = \{i\gamma_5, i\gamma_\mu\}$  for  $J = \{P, V\}$ .

We have solved Eq.(3) within a variational-integral approach and have found that the following values of free parameters:

$$\begin{aligned} \alpha_s &= 0.186, & \Lambda &= 730 \text{ MeV}, & m_{\{u, d\}} &= 170 \text{ MeV}, \\ m_s &= 188 \text{ MeV}, & m_c &= 646 \text{ MeV}, & m_b &= 4221 \text{ MeV}. \end{aligned} \quad (5)$$

are optimal for the meson ground states, orbital and radial excitations [6]. Our estimates for the  $P$  and  $V$  meson masses in the ground state (Fig.1) were in good agreement with the experimental data [17] and have a small experimental error ( $1 \div 5$  per cent) in a wide range  $\sim 140 \div 9500$  MeV.

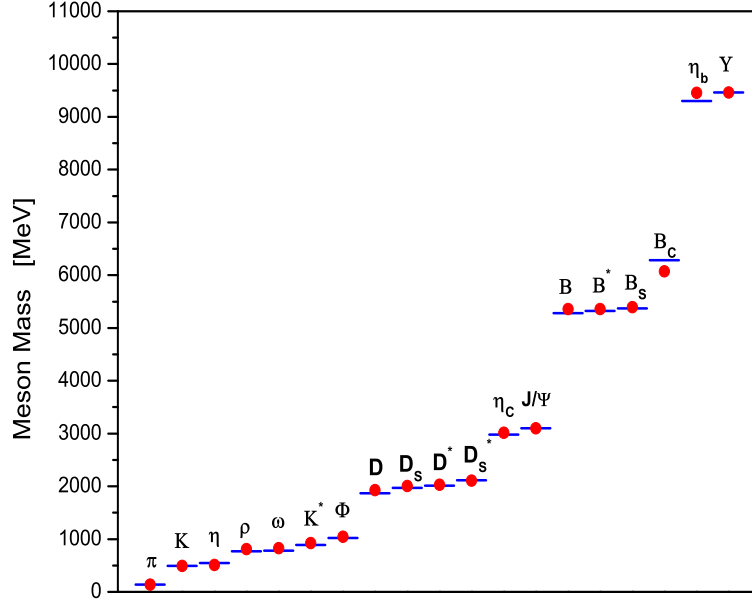


Figure 1: Estimated meson masses (dots) compared with experimental data (dashes).

The lowest-state (scalar) glueball mass has been derived analytically [6]:

$$M_G^2 = 2\Lambda^2 \ln \left( \frac{\alpha_{upp}}{\alpha_s} \right) = (1745 \text{ MeV})^2, \quad \alpha_{upp} = \frac{2\pi(2 + \sqrt{3})^2}{27}. \quad (6)$$

Our estimate (6) is close to the result  $M_G = 1710 \pm 100$  MeV due to the quantized knot soliton model [18] and quenched lattice estimate [19] and in agreement with the predictions expecting non- $q\bar{q}$  scalar object in the range  $\sim 1600 \div 1800$  MeV [20, 21].

The estimated value  $\alpha_s = 0.186$  coincides with the latest experimental data  $\alpha_{s,(expr)} \approx 0.10 \div 0.35$  [17] and the prediction of the quenched theory  $\alpha_{s,(quench)} \approx 0.195$  [22]. Note, this relatively weak coupling justifies the use of the one-gluon exchange mode in our consideration. By analysing the glueball mass formula (6) one obtains the restriction  $\alpha_s < \alpha_{upp} \approx 3.24124$  that coincides with the two-loop value of the freezing coupling constant [23].

A rough estimate of the hadronization distance

$$r_{hadr}^2 \sim \frac{\int d^4x \, x^2 \, D(x)}{\int d^4x \, x \, D(x)} = \frac{8}{\Lambda^2} \sim \left( \frac{1}{250 \text{ MeV}} \right)^2$$

shows that the confinement radius  $r_{conf} \simeq 1/\Lambda$  and  $r_{hadr}$  are comparable values.

Below, we extend our consideration to the light meson decay constants.

### 3 Light Meson Decay Constants

The decay constant  $f_P$  is defined by the matrix element:

$$if_P p_\mu = \langle 0 | J_\mu(0) | U(p) \rangle, \quad (7)$$

where  $J_\mu$  is the axial vector current and  $U(p)$  is the normalized state vector.

There have been many attempts to extract  $f_P$  from meson spectroscopy. Particularly, the decay constant  $f_{\pi^+}$  for  $\pi^+$ -meson is determined from the combined rate for  $\pi^+ \rightarrow \mu^+ \nu_\mu + \mu^+ \nu_\mu \gamma$ . With the recent experimental data one obtains [17]:

$$f_\pi^{exp} = 130.7 \pm 0.1 \pm 0.36 \text{ MeV}, \quad f_K^{exp} = 159.8 \pm 1.4 \pm 0.44 \text{ MeV}, \quad f_K^{exp}/f_\pi^{exp} \simeq 1.22 \pm 0.02.$$

Recent lattice QCD model with exact chiral symmetry estimates [24]:

$$f_K^{latt} = 152 \pm 6 \pm 10 \text{ MeV}.$$

The unquenched lattice QCD calculation predicts [25]

$$(f_K/f_\pi)_{unquen} = 1.198 \pm 0.019.$$

Following our approach, we estimate  $f_P$  as follows:

$$if_P p_\mu = \frac{g}{6} \int \frac{dk}{(2\pi)^4} \int dx e^{-ikx} U(x) \sqrt{D(x)} \text{Tr} \left\{ i\gamma_5 \tilde{S}(\hat{k} + \mu_1 \hat{p}) i\gamma_5 \gamma_\mu \tilde{S}(\hat{k} - \mu_2 \hat{p}) \right\}.$$

It is reasonable to choose a normalized trial function [6]:

$$U(x) \sim \sqrt{D(x)} \exp \left\{ -a\Lambda^2 x^2/4 \right\}, \quad \int dx |U(x)|^2 = 1, \quad a > 0. \quad (8)$$

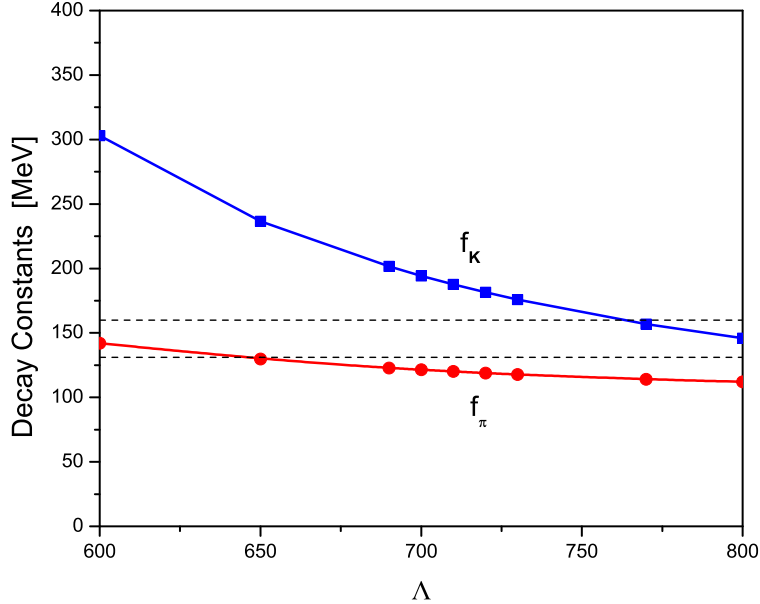


Figure 2: Evolution of  $f_\pi$  and  $f_K$  with  $\Lambda$ . Dashed lines depict the experimental data.

Omitting details of intermediate calculations write the final expression

$$f_P = \frac{\sqrt{\alpha_s} (m_1 + m_2)}{12\pi^{3/2}(m_1/\Lambda)^2(m_2/\Lambda)^2} \exp \left\{ \frac{m_1^2 + m_2^2}{2\Lambda^2} \left( \frac{M_P^2}{(m_1 + m_2)^2} - 1 \right) \right\} \quad (9)$$

$$\cdot \max_{1/2 < c < 1} \left\{ (3c - 1)(1 - c) \frac{2m_1m_2 + c(m_1 - m_2)^2}{(m_1 + m_2)^2} \exp \left[ -\frac{cM_P^2}{4\Lambda^2} \left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 \right] \right\}.$$

The solution of Eq.(9) for different values of  $\Lambda$  is plotted in Fig.2 in comparison with the experimental data [17]. In the region  $650 \div 760\text{MeV}$  we underestimate  $f_\pi^{exp}$  and overestimate  $f_K^{exp}$ . Evolutions of  $f_K/f_\pi$  and  $\alpha_s$  with  $\Lambda$  are depicted in Fig.3.

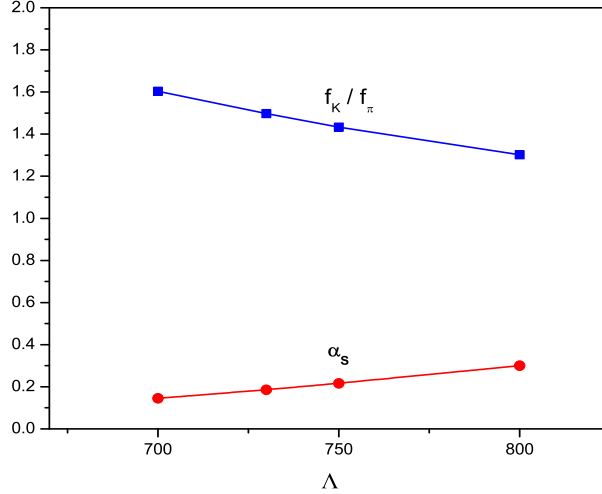


Figure 3: The ratio  $f_K/f_\pi$  and the coupling strength  $\alpha_s$  versa the confinement scale  $\Lambda$ .

For consistency, we use the parameters (5) and obtain:

$$f_\pi = 118\text{MeV}, \quad f_K = 176\text{MeV}.$$

Our estimates lie near to the latest data [17, 24, 25].

In conclusion, we have considered a relativistic quantum field model of interacting quarks and gluons under the analytic confinement and solve the Bethe-Salpeter equation in the one-gluon exchange approximation for the quark-antiquark ( $q\bar{q}'$ ) bound states. Despite the simplicity, this model resulted in a quite reasonable sight to the underlying physical principles of the hadronization mechanism and gave an accurate description of the ground states, orbital ( $\ell > 0$ ) and radial ( $n_r > 0$ ) excitations of mesons in a wide range of mass (up to 9.5 GeV) and acceptable values of the light meson decay constants  $\{f_\pi, f_K\}$ . Our model provides a solid framework to compute the meson spectra, the lowest glueball state and decay constants of light mesons from the basic principles of QCD and QFT.

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